

ON QUASI-INVERSIONS

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ABSTRACT. Given a bounded domain $D \subset \mathbb{R}^n$ strictly starlike with respect to $0 \in D$, we define a quasi-inversion w.r.t. the boundary ∂D . We show that the quasi-inversion is bi-Lipschitz w.r.t. the chordal metric if and only if every "tangent line" of ∂D is far away from the origin. Moreover, the bi-Lipschitz constant tends to 1, when ∂D approaches the unit sphere in a suitable way. For the formulation of our results we use the concept of the α -tangent condition due to F. W. Gehring and J. Väisälä (Acta Math. 1965). This condition is shown to be equivalent to the bi-Lipschitz and quasiconformal extension property of what we call the polar parametrization of ∂D . In addition, we show that the polar parametrization, which is a mapping of the unit sphere onto ∂D , is bi-Lipschitz if and only if D satisfies the α -tangent condition.

KEYWORDS. Quasi-inversion, starlike domain, α -tangent condition

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1. INTRODUCTION

Möbius transformations in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ can be defined as mappings of the form $F = h_1 \circ \cdots \circ h_m$, where each h_j , $j = 1, \dots, m$, is either a reflection in a hyperplane of \mathbb{R}^n or an inversion in a sphere $S^{n-1}(a, r) = \{x \in \mathbb{R}^n : |x - a| = r\}$, a mapping of the form [Ah, B]

$$(1.1) \quad x \mapsto h(x) = a + \frac{r^2(x - a)}{|x - a|^2}, \quad x \in \mathbb{R}^n \setminus \{a\},$$

and $h(a) = \infty$, $h(\infty) = a$, where $a \in \mathbb{R}^n$ and $r > 0$. Möbius transformations have an important role in the study of the hyperbolic geometry of the unit ball \mathbb{B}^n or of the upper half space $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ because the hyperbolic metric remains invariant under Möbius automorphisms of the corresponding space. This invariance property immediately follows from the characterizing property of Möbius transformations [B, p. 32, Theorem 3.2.7]: the absolute ratio of every quadruple stays invariant under a Möbius transformation f

$$(1.2) \quad |f(a), f(b), f(c), f(d)| = |a, b, c, d|.$$

One of the basic facts is the distance formula for the mapping h in (1.1) [Ah],[B, 3.1.5], [Vu, 1.5]:

$$(1.3) \quad |h(x) - h(y)| = \frac{r^2|x - y|}{|x - a||y - a|}.$$

Because Möbius transformations are conformal maps they also have a role in the theory of quasiconformal maps which is the motivation of the present study. Our starting point is to define a quasi-inversion with respect to the boundary $\mathcal{M} = \partial D$ of a bounded domain $D \subset \mathbb{R}^n$, strictly starlike w.r.t. the origin. Given a point $x \in \mathbb{R}^n \setminus \{0\}$, consider the ray

$L_x = \{tx : t > 0\}$ and write $r_x = |w|$ where w is the unique point in $L_x \cap \mathcal{M}$. Then $r_x = r_{sx}$ for all $s > 0$. We define the quasi-inversion in \mathcal{M} by

$$(1.4) \quad x \mapsto f_{\mathcal{M}}(x) = \frac{r_x^2}{|x|^2}x, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and $f_{\mathcal{M}}(0) = \infty$, $f_{\mathcal{M}}(\infty) = 0$ (e.g. see Figure 1). Then $f_{\mathcal{M}}(x) = x$ for $x \in \mathcal{M}$ and $f_{\mathcal{M}}(D) = \mathbb{R}^n \setminus \overline{D}$.

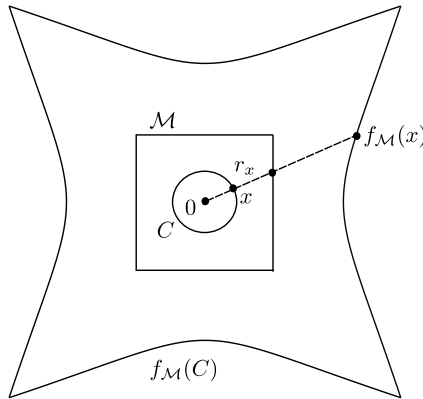


FIGURE 1. The image $f_{\mathcal{M}}(C)$ of the circle $C = S^1(0, 1/2)$ under the quasi-inversion $f_{\mathcal{M}}$. Here \mathcal{M} is the boundary of the square centered at 0 with side length 2.

Obviously the quasi-inversion in $S^{n-1}(0, r)$ coincides with the inversion h in (1.1) when $a = 0$. Therefore it is a natural question to study which properties of inversions hold for quasi-inversions, too.

Perhaps the simplest question is to investigate under which conditions on ∂D we have a counterpart of the identity (1.3) in the form of an inequality, with a constant depending on ∂D . This is the content of our first main result, formulated as Theorem 5.3. Secondly, it is a basic fact that the inversion in (1.1) with $a = 0$ and $r = 1$ is a 1-bi-Lipschitz (isometry) mapping w.r.t. the chordal metric q (see Section 2). The result here for quasi-inversions is formulated as Theorem 5.9. Third, because the inversion h in $\partial \mathbb{B}^n$ transforms the points $x, y \in \mathbb{B}^n$ to $h(x), h(y) \in \mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ with the equal hyperbolic distances $\rho_{\mathbb{B}^n}(x, y) = \rho_{\mathbb{R}^n \setminus \overline{\mathbb{B}^n}}(h(x), h(y))$ we may look for a similar result in the quasi-inversion case. This question requires finding a suitable counterpart of the hyperbolic metric for a strictly starlike domain D and its complementary domain $\mathbb{R}^n \setminus \overline{D}$. Presumably several metrics could be used here. In our third main result we generalize this to the case of a Möbius invariant metric introduced by P. Seittenranta, see Theorem 5.10. Moreover, several other results will be proved. It should be pointed out that the idea of quasi-inversion (without the name) was used by M. Fait, J. G. Krzyż, and J. Zygmunt in [FKZ]. Some of the notions we use are close to the study of F. W. Gehring and J. Väisälä [GV] concerning quasiconformal mapping problems in \mathbb{R}^3 . We refine some of these results and also a later result by O. Martio and U. Srebro [MS].

For a nonempty subset $\mathcal{A} \subset \mathbb{R}^n \setminus \{0\}$, the radial projection $\Pi : \mathcal{A} \rightarrow S^{n-1}$ is defined by

$$(1.5) \quad \Pi(x) = x/|x|.$$

Let $\mathcal{M} \subset \mathbb{R}^n$ be now the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin. Then the inverse function of Π is denoted by $\varphi : S^{n-1} \rightarrow \mathcal{M}$. Further, we define the radial

extension of φ by

$$(1.6) \quad \varphi_a : x \mapsto |x|^a \varphi(x/|x|), \quad x \in \mathbb{R}^n \setminus \{0\},$$

and $\varphi_a(0) = 0$, $\varphi_a(\infty) = \infty$, where $a > 0$. Note that for $\mathcal{M} = S^{n-1}$ this mapping is the standard radial stretching [Va, p. 49]. The main result of Section 3 is Theorem 3.15, which states roughly that the radial projection of \mathcal{M} is bi-Lipschitz if and only if the domain bounded by \mathcal{M} satisfies the α -tangent condition, and the bi-Lipschitz constant depends only on α and the distance from origin to \mathcal{M} . Our results imply the unexpected fact that the global bi-Lipschitz constant of the radial projection of a hyper-surface onto the unit sphere is equal to the maximal value of the local bi-Lipschitz constant. The main result of Section 4 is Theorem 4.14 which says that φ_1 is bi-Lipschitz if and only if the domain bounded by \mathcal{M} satisfies the α -tangent condition, with an explicit bi-Lipschitz constant. Moreover, φ_a is quasiconformal if and only if the domain bounded by \mathcal{M} satisfies the α -tangent condition. Further, we give an explicit constant of quasiconformality in terms of the angle $\alpha \in (0, \pi/2]$ and a . We finish the paper by proving that the quasi-inversion in \mathcal{M} is a K -quasiconformal mapping with $K = \cot^2 \frac{\alpha}{2}$, see Theorem 5.11 in Section 5.

2. PRELIMINARIES

Let $B^n(x, r)$ be the ball centered at x with radius $r > 0$ and $S^{n-1}(x, r)$ its boundary sphere. We abbreviate $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = S^{n-1}(0, r)$, $\mathbb{B}^n = B^n(1)$, $S^{n-1} = S^{n-1}(1)$. Let $[a, b]$ be the segment with endpoints a, b .

2.1. Dilatations. Let D, D' be subdomains of \mathbb{R}^n and $f : D \rightarrow D'$ be a differentiable homeomorphism and denote its Jacobian by $J(x, f)$, $x \in D$. If $x \in D$ and $J(x, f) \neq 0$, then the derivative of f at $x \in D$ is a bijective linear mapping $f'(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and we denote

$$(2.2) \quad H_I(f'(x)) = \frac{|J(x, f)|}{\lambda_f(x)^n}, \quad H_O(f'(x)) = \frac{\Lambda_f(x)^n}{|J(x, f)|}, \quad H(f'(x)) = \frac{\Lambda_f(x)}{\lambda_f(x)},$$

where

$$\Lambda_f(x) := \max\{|f'(x)h| : |h| = 1\} \quad \text{and} \quad \lambda_f(x) := \min\{|f'(x)h| : |h| = 1\}.$$

Sometimes instead of $|\Lambda_f(x)|$ we use notation $|f'(x)|$, to denote the norm of the matrix $A = f'(x)$. If $\lambda_1^2 \leq \dots \leq \lambda_n^2$ ($\lambda_i > 0, i = 1, 2, \dots, n$) are eigenvalues of the symmetric matrix AA^t where A^t is the adjoint of A , then we have the following well-known formulas

$$(2.3) \quad |J(x, f)| = \prod_{k=1}^n \lambda_k, \quad \Lambda_f(x) = \lambda_n, \quad \lambda_f(x) = \lambda_1.$$

By (2.2) and (2.3), we arrive at the following simple inequalities [Va, 14.3]

$$(2.4) \quad H(f'(x)) \leq \min\{H_I(f'(x)), H_O(f'(x))\} \leq H(f'(x))^{n/2}.$$

$$(2.5) \quad H(f'(x))^{n/2} \leq \max\{H_I(f'(x)), H_O(f'(x))\} \leq H(f'(x))^{n-1}.$$

The quantities

$$K_I(f) = \sup_{x \in D} H_I(f'(x)), \quad K_O(f) = \sup_{x \in D} H_O(f'(x))$$

are called the inner and outer dilatation of f , respectively. The maximal dilatation of f is

$$K(f) = \max\{K_I(f), K_O(f)\}.$$

2.6. Quasiconformal mappings. In the literature, see e.g. [C], we can find various definitions of quasiconformality which are equivalent. The following analytic definition for quasiconformal mappings is from [Va, Theorem 34.6]: a homeomorphism $f : D \rightarrow D'$ is C -quasiconformal if and only if the following conditions are satisfied: (i) f is ACL; (ii) f is differentiable a.e.; (iii) $\Lambda_f(x)^n/C \leq |J(x, f)| \leq C\lambda_f(x)^n$ for a.e. $x \in D$. By [Va, Theorem 34.4], if f satisfies the conditions (i), (ii) and $J(x, f) \neq 0$ a.e., then

$$K_I(f) = \operatorname{ess\,sup}_{x \in D} H_I(f'(x)), \quad K_O(f) = \operatorname{ess\,sup}_{x \in D} H_O(f'(x)).$$

Hence (iii) can be written as $K(f) \leq C$ which by (2.5) is equivalent to

$$(2.7) \quad H(f'(x)) \leq K \text{ for a.e. } x \in D.$$

Here the constant $K \leq C^{2/n}$. In this paper we say that a quasiconformal mapping $f : D \rightarrow D'$ is K -quasiconformal if K satisfies (2.7).

It is important to notice that f is K -quasiconformal if and only if f^{-1} is K -quasiconformal and that the composition of K_1 and K_2 quasiconformal mappings is $K_1 K_2$ -quasiconformal. (It is well-known that this also holds for K -quasiconformality in Väisälä's sense, see [Va, Corollary 13.3, Corollary 13.4]).

Recall that for the case of planar differentiable mapping f , we have

$$(2.8) \quad \Lambda_f(z) = |f_z| + |f_{\bar{z}}| = |f'(z)|,$$

$$(2.9) \quad \lambda_f(z) = ||f_z| - |f_{\bar{z}}|| = \frac{1}{|(f^{-1})'(z)|}.$$

Hence the condition (2.7) can be written as

$$|\mu_f(z)| \leq k \text{ a.e. on } D \text{ where } k = \frac{K-1}{K+1}$$

and $\mu_f(z) = f_{\bar{z}}/f_z$ is the complex dilatation of f . Sometimes instead of K -quasiconformal we write k -quasiconformal.

2.10. Lipschitz mappings. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \rightarrow Y$ be continuous. We say that f is L -Lipschitz, if

$$(2.11) \quad d_Y(f(x), f(y)) \leq L d_X(x, y), \text{ for } x, y \in X,$$

and L -bi-Lipschitz, if f is a homeomorphism and

$$(2.12) \quad d_X(x, y)/L \leq d_Y(f(x), f(y)) \leq L d_X(x, y), \text{ for } x, y \in X.$$

A 1-bi-Lipschitz mapping is called an isometry. The smallest L for which (2.11) holds will be denoted by $\operatorname{Lip}(f)$.

Let $f : D \rightarrow fD$ be a Lipschitz map, where $D, fD \subset \mathbb{R}^n$ and D is convex. By the Mean value theorem, we have the following simple fact

$$(2.13) \quad \operatorname{Lip}(f) = \operatorname{ess\,sup}_{x \in D} |f'(x)|.$$

Recall that Lipschitz maps are a.e. differentiable by the Rademacher-Stepanov theorem [Va, p. 97].

2.14. Starlike domains. A bounded domain $D \subset \mathbb{R}^n$ is said to be *strictly starlike w.r.t. the point a* if each ray emanating from a meets ∂D at exactly one point.

2.15. α -tangent condition. Suppose $D \subset \mathbb{R}^n$ is a strictly starlike domain w.r.t. the origin and $x \in \partial D$. For each $z \in \partial D$, $z \neq x$, we let $\alpha(z, x)$ denote the acute angle which the segment $[z, x]$ makes with the ray from 0 through x , and we define

$$\alpha(x) = \liminf_{z \rightarrow x} \alpha(z, x) \in [0, \pi/2].$$

If ∂D has a tangent hyperplane at x whose normal forms an acute angle θ with the ray from 0 through x , then

$$\alpha(x) = \pi/2 - \theta.$$

We say a domain D satisfies the α -tangent condition if for every $x \in \partial D$ we have $\alpha(x) \geq \alpha \in (0, \pi/2]$.

F. W. Gehring and J. Väisälä [GV, Theorem 5.1] studied the dilatations of a homeomorphism in terms of the above α -tangent condition (without the name).

2.16. β -cone condition (Martio and Srebro [MS]). Let $D \subset \mathbb{R}^n$ be a bounded domain with $0 \in D$ and let $\beta \in (0, \pi/4]$. We say that D satisfies the β -cone condition if the open cone

$$C(x, \beta) := \{z \in \mathbb{R}^n : |z - x| < |x|, \langle z - x, x \rangle > |x - z||x| \cos \beta\}$$

with vertex x and central angle β lies in D whenever $x \in \partial D$. Note that if D satisfies the β -cone condition, then D is strictly starlike.

Proposition 2.17. *A domain D satisfies the β -cone condition if and only if it satisfies the α -tangent condition.*

Proof. It is obvious that D satisfies the α -tangent condition with $\alpha = \beta$ if D satisfies the β -cone condition.

We want to show that if D satisfies the α -tangent condition, then D satisfies the β -cone condition. First, fix $x \in \partial D$, we prove that there exists a constant $\beta_x \in (0, \pi/4]$, such that $C(x, \beta_x) \subset D$. If not, then for all $n \in \mathbb{N}$, there exists a sequence $a_n \in C(x, 1/n) \cap \partial D$. Since ∂D is compact there is a subsequence still denoted by $\{a_n\}$ converging to a limiting point $a \in (0, x] \cap \partial D$. If $a = x$, then $\alpha(x) = \liminf_{n \rightarrow \infty} \alpha(a_n, x) = 0$ which contradicts the α -tangent condition. If $a \in (0, x)$, a contradiction with the starlikeness of D follows.

Next, we prove that there exists a uniform constant $\beta \in (0, \pi/4]$ for all $x \in \partial D$, such that $C(x, \beta) \subset D$. Suppose not, then for all $n \in \mathbb{N}$, there exists $x_n \in \partial D$ such that $C(x_n, 1/n) \cap \partial D \neq \emptyset$. Since ∂D is compact there is a subsequence $x_{n_k} \rightarrow x_0 \in \partial D$, as $k \rightarrow \infty$. For this x_0 there is no open cone $C(x_0, \beta_{x_0}) \subset D$ which is a contradiction. \square

Remark 2.18. It can be proved that if the domain satisfies the β -cone condition almost everywhere (or on some dense subset of the boundary), then it satisfies the β -cone condition everywhere. On the other hand there exists a domain satisfying the α -tangent condition almost everywhere but not everywhere. For example, consider the domain D bounded by the graph of the Cantor step function \mathcal{C} and the following segments $[T, A]$, $[A, B]$, $[B, C]$, and $[C, O]$, where $T = (1, 1)$, $A = (2, 1)$, $B = (2, -1)$, $C = (0, -1)$, $O = (0, 0)$ (Figure 2). The domain D is strictly starlike w.r.t. the point $S = (1, -1/2)$ and satisfies the α -tangent condition but not the β -cone condition. A point where the cone condition is ruined is $T = (1, 1)$ together with some points in its neighborhood. The size of the neighborhood depends on a given positive number β .

2.19. The chordal metric. The chordal metric is defined by

$$(2.20) \quad \begin{cases} q(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x, y \neq \infty, \\ q(x, \infty) = \frac{1}{\sqrt{1+|x|^2}}, & x \neq \infty. \end{cases}$$

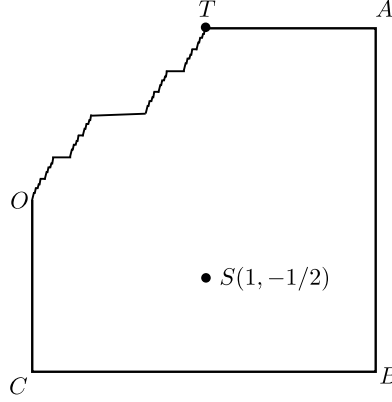


FIGURE 2. A domain satisfies the α -tangent condition almost everywhere with $\alpha = \arctan 1/2$, but not everywhere.

2.21. Möbius metric (Seittenranta [S2]). Let G be an open subset of \mathbb{R}^n with $\text{card } \partial G \geq 2$. For all $x, y \in G$, the Möbius (or absolute ratio) metric δ_G is defined as

$$\delta_G(x, y) = \log(1 + \sup_{a, b \in \partial G} |a, x, b, y|), \quad |a, x, b, y| = \frac{|a - b||x - y|}{|a - x||b - y|}.$$

It is a well-known basic fact that δ_G agrees with the hyperbolic metric both in the case of the unit ball as well as in the case of the half space (cf. [Vu, 8.39]).

2.22. Ferrand's metric. Let $G \subset \mathbb{R}^n$ be a domain with $\text{card } \partial G \geq 2$. We define a continuous density function

$$w_G(x) = \sup_{a, b \in \partial G} \frac{|a - b|}{|x - a||x - b|}, \quad x \in G \setminus \{\infty\},$$

and a metric σ in G ,

$$\sigma_G(x, y) = \inf_{\gamma \in \Gamma} \int_{\gamma} w_G(t) |dt|,$$

where Γ is the family of all rectifiable curves joining x and y in G .

Proposition 2.23. *The chordal metric q is invariant under the inversion in the unit sphere. The Möbius metric δ_G and Ferrand's metric σ_G are Möbius-invariant.*

Lemma 2.24. [S2, Theorem 3.16] *Let $G, G' \subsetneq \mathbb{R}^n$ be open sets and $f : G \rightarrow G'$ an L -bi-Lipschitz map w.r.t. the Euclidean metric. Then f is an L^4 -bi-Lipschitz map w.r.t. the Möbius metric.*

Lemma 2.25. [S2, Theorem 3.18] *Let $G, G' \subsetneq \mathbb{R}^n$ be domains and $f : G \rightarrow G'$ an L -bi-Lipschitz map w.r.t. the Euclidean metric. Then f is an L^4 -bi-Lipschitz map w.r.t. Ferrand's metric.*

Without a proof, the following lemma is given in [S1, 1.34].

Lemma 2.26. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(0) = 0$ and $f(\infty) = \infty$.*

(1) *If $f|_{\mathbb{R}^n}$ is an L -bi-Lipschitz map w.r.t. the Euclidean metric, then f is an L^3 -bi-Lipschitz map w.r.t. the chordal metric.*

(2) *If f is an L -bi-Lipschitz map w.r.t. the chordal metric, then $f|_{\mathbb{R}^n}$ is an L^3 -bi-Lipschitz map w.r.t. the Euclidean metric.*

Proof. (1) If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an L -bi-Lipschitz mapping w.r.t. the Euclidean metric and $f(0) = 0$, then

$$(2.27) \quad \frac{1}{L}|x - y| \leq |f(x) - f(y)| \leq L|x - y|$$

and

$$(2.28) \quad \frac{1}{L}|x| \leq |f(x)| \leq L|x|.$$

For $x, y \in \mathbb{R}^n$ and $x \neq y$, we have

$$\frac{q(f(x), f(y))}{q(x, y)} = \frac{|f(x) - f(y)|}{|x - y|} \cdot \sqrt{\frac{1 + |x|^2}{1 + |f(x)|^2}} \cdot \sqrt{\frac{1 + |y|^2}{1 + |f(y)|^2}},$$

by (2.27) and (2.28), and hence

$$\frac{q(f(x), f(y))}{q(x, y)} \leq L^3.$$

If $y = \infty$ and $x \neq y$, then

$$\frac{q(f(x), f(\infty))}{q(x, \infty)} = \sqrt{\frac{1 + |x|^2}{1 + |f(x)|^2}} \leq L.$$

Applying the above argument to f^{-1} , we easily get

$$\frac{q(f^{-1}(x), f^{-1}(y))}{q(x, y)} \leq L^3$$

and hence

$$\frac{q(f(x), f(y))}{q(x, y)} \geq \frac{1}{L^3}.$$

(2) If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an L -bi-Lipschitz mapping w.r.t. the chordal metric and $f(\infty) = \infty$, then

$$(2.29) \quad \frac{1}{L}q(x, y) \leq q(f(x), f(y)) \leq Lq(x, y)$$

and

$$(2.30) \quad \frac{1}{L}q(x, \infty) \leq q(f(x), f(\infty)) \leq Lq(x, \infty).$$

For $x, y \in \mathbb{R}^n$ and $x \neq y$, we have

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{q(f(x), f(y))}{q(x, y)} \cdot \frac{q(x, \infty)}{q(f(x), f(\infty))} \cdot \frac{q(y, \infty)}{q(f(y), f(\infty))},$$

by (2.29) and (2.30), and hence

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L^3.$$

Applying the above argument to f^{-1} , we finally get

$$\frac{|f(x) - f(y)|}{|x - y|} \geq \frac{1}{L^3}.$$

This completes the proof. \square

Remark 2.31. The authors are indebted to T. Sugawa for pointing out a connection between the α -tangent condition and strongly starlike plane domain of order $1 - (2/\pi)\alpha$ in the sense of Brannan-Kirwan and Stankiewicz. See [Su] for the details.

3. THE RADIAL PROJECTION TO THE UNIT SPHERE

Let $\text{dist}(\mathcal{A}, a)$ be the distance from the point a to the set \mathcal{A} . The radial projection in (1.5) maps a point $z \in \mathbb{R}^n \setminus \{0\}$ to

$$\Pi(z) = z^*; \quad z^* = z/|z|.$$

In this section, we mainly study the Lipschitz properties of this projection.

J. Luukkainen and J. Väisälä obtained

$$|\Pi(x) - \Pi(y)| \leq |x - y| / \sqrt{|x||y|},$$

for $x, y \in \mathbb{R}^n \setminus \{0\}$ ([LV, Lemma 2.12]). The following lemma improves this inequality.

Lemma 3.1. *For two distinct points $x, y \in \mathbb{R}^n \setminus \{0\}$, there holds*

$$(3.2) \quad \frac{|x - y|}{|x^* - y^*|} \geq \frac{|x| + |y|}{2},$$

the equality is attained if and only if $|x| = |y|$.

Proof. Without loss of generality we may assume that $0, x$ and y are three distinct complex numbers. Let $x = pe^{is}$ and $y = qe^{it}$ where $p, q > 0$, $s, t \in [0, 2\pi]$, and $s \neq t$. Then

$$\left(\frac{|x - y|}{|x^* - y^*|} \right)^2 - \left(\frac{|x| + |y|}{2} \right)^2 = \frac{1}{4}(p - q)^2 \cot^2 \left(\frac{t - s}{2} \right) \geq 0,$$

which implies the inequality (3.2). The equality case is clear. \square

By Lemma 3.1 we immediately have the following corollaries.

Corollary 3.3. *Let \mathcal{A} be any nonempty subset of \mathbb{R}^n with $\text{dist}(\mathcal{A}, 0) > 0$. For two distinct points $x, y \in \mathcal{A}$ we have*

$$\frac{|x - y|}{|x^* - y^*|} \geq \text{dist}(\mathcal{A}, 0).$$

Corollary 3.4. *Let \mathcal{A} be any nonempty subset of \mathbb{R}^n with $\text{dist}(\mathcal{A}, 0) > 0$. Let $\Pi : \mathcal{A} \rightarrow S^{n-1}$ be the radial projection. Then Π is Lipschitz continuous and*

$$(3.5) \quad \text{Lip}(\Pi) \leq \frac{1}{\text{dist}(\mathcal{A}, 0)}.$$

Our next aim is to study the equality case in (3.5), under suitable conditions on \mathcal{A} . Fix a connected closed set $\mathcal{A} \subset \mathbb{R}^n \setminus \{0\}$ with $\text{dist}(\mathcal{A}, 0) > 0$. Let $N(\mathcal{A}) = \{h \in \mathcal{A} : \text{dist}(\mathcal{A}, 0) = |h|\}$. We say that \mathcal{A} is **admissible** if there exist two sequences of distinct points $\{x_k\}, \{y_k\} \subset \mathcal{A}$ tending to two points h and h' , respectively, $h, h' \in N(\mathcal{A})$ and satisfying one of the following conditions:

- (i) $h \neq h'$;
- (ii) $h = h'$ and

$$(3.6) \quad |x_k| - |y_k| = o(\angle(x_k, 0, y_k)), \quad k \rightarrow \infty.$$

Example 3.7. The important examples of admissible subsets are the boundaries of domains in \mathbb{R}^n strictly starlike w.r.t. the origin. Let \mathcal{M} be one of these boundaries. Because \mathcal{M} is compact, we may assume that $h \in \mathcal{M}$ which is one of the closest points from the origin. In fact there exist two sequences of distinct points $\{x_k\}, \{y_k\} \subset \mathcal{M}$ satisfying $|x_k| = |y_k|$, which converge to the same point h and satisfy (3.6). We consider \mathcal{M} in two cases. The existence of such sequences is trivial if \mathcal{M} coincides with a sphere centered at origin. Otherwise we consider the function $\phi : \mathcal{M} \rightarrow \mathbb{R}^+$ by $\phi(x) = |x|$. Then there is a $\delta > 0$ such that $[|h|, |h| + \delta] \subset \phi(\mathcal{M})$. This means that for $t \in (|h|, |h| + \delta)$ there exists $x \in \mathcal{M}$ such that $\phi(x) = t$. The subset $\phi^{-1}(y) = \mathcal{M} \cap S^{n-1}(0, t)$ of \mathcal{M} contains an element y different from x . This case is trivial for $n \geq 3$, since $n - 1 \geq 2$, and deleting a point, we cannot ruin the connectivity of the set. If $n - 1 = 1$, then we use an additional argument that \mathcal{M} is a closed curve, which means that it cannot be separated by deleting only one point. Thus for all dimensions $n \geq 2$ and all $\mathcal{M} \subset \mathbb{R}^n$ as above, there exist different sequences of points $\{x_k\}, \{y_k\} \subset \mathcal{M}$ satisfying $|x_k| = |y_k|$. This means that they satisfy (3.6).

We offer a counterexample.

Example 3.8. Let $\mathcal{A} \subset \mathbb{R}^2$ be the union of the unit circle S^1 and the interval $[1/2, 1]$. Then $h = 1/2 \in N(\mathcal{A})$, and if two sequences of distinct points $\{x_k\}, \{y_k\} \subset \mathcal{A}$ converge to $1/2$, then $\angle(x_k, 0, y_k) = 0$ as $k \rightarrow \infty$, which means that they do not satisfy (3.6). Further, we consider the radial projection $\Pi : \mathcal{A} \rightarrow S^1$. For two distinct points $x, y \in \mathcal{A}$, we have

$$\frac{|\Pi(x) - \Pi(y)|}{|x - y|} = \begin{cases} 1, & x, y \in S^1, \\ 0, & x, y \in [1/2, 1]. \end{cases}$$

If $x \in [1/2, 1]$ and $y \in S^1 \setminus \{1\}$, by symmetry we may assume that $y = e^{i\theta}$ ($0 < \theta \leq \pi$). Then

$$\max_x \left\{ \frac{|\Pi(x) - \Pi(y)|}{|x - y|} \right\} = \begin{cases} \frac{|e^{i\theta} - 1|}{\sin \theta}, & \theta \in (0, \pi/3], \\ \frac{|e^{i\theta} - 1|}{|e^{i\theta} - 1/2|}, & \theta \in (\pi/3, \pi]. \end{cases}$$

By calculation, we have

$$\frac{|e^{i\theta} - 1|}{\sin \theta} = \sqrt{\frac{2}{1 + \cos \theta}} \leq 2/\sqrt{3} \text{ if } \theta \in (0, \pi/3]$$

and

$$\frac{|e^{i\theta} - 1|}{|e^{i\theta} - 1/2|} = \sqrt{2 - \frac{2}{5 - 4 \cos \theta}} \leq 4/3 \text{ if } \theta \in (\pi/3, \pi].$$

Therefore,

$$\text{Lip}(\Pi) = 4/3 < 2 = 1/\text{dist}(\mathcal{A}, 0).$$

Lemma 3.9. Let \mathcal{A} be a nonempty subset of \mathbb{R}^n with $\text{dist}(\mathcal{A}, 0) > 0$. Let $\Pi : \mathcal{A} \rightarrow S^{n-1}$ be the radial projection. Then $\text{Lip}(\Pi) = 1/\text{dist}(\mathcal{A}, 0)$ if and only if \mathcal{A} is admissible.

Proof. Assume that \mathcal{A} is admissible and we consider two cases.

Case 1. There exist two sequences of distinct points $\{x_k\}, \{y_k\} \subset \mathcal{A}$ converging to two different points h, h' , respectively, with $|h| = |h'| = \text{dist}(\mathcal{A}, 0)$, i.e. $\lim_{k \rightarrow \infty} x_k = h$ and $\lim_{k \rightarrow \infty} y_k = h'$. Then we have

$$\lim_{k \rightarrow \infty} \frac{|x_k^* - y_k^*|}{|x_k - y_k|} = \frac{|h^* - h'^*|}{|h - h'|} = \frac{|h^*|}{|h|} = \frac{1}{\text{dist}(\mathcal{A}, 0)}.$$

Case 2. There exist two sequences of distinct points $\{x_k\}, \{y_k\} \subset \mathcal{A}$ converging to the same point h , i.e.

$$\lim_{k \rightarrow \infty} x_k = h = \lim_{k \rightarrow \infty} y_k.$$

In the notation of the proof of Lemma 3.1, we assume that the points $0, x_k, y_k$ belong to the complex plane, too. Let $x_k = p_k e^{is_k}$, $y_k = q_k e^{it_k}$ where $p_k, q_k > 0$, $s_k, t_k \in [0, 2\pi]$, and $s_k \neq t_k$. By (3.6) we have

$$(3.10) \quad \left(\frac{|x_k - y_k|}{|x_k^* - y_k^*|} \right)^2 - \left(\frac{|x_k| + |y_k|}{2} \right)^2 = \frac{1}{4} (p_k - q_k)^2 \cot^2 \left(\frac{t_k - s_k}{2} \right) \rightarrow 0, \quad k \rightarrow \infty,$$

which implies

$$\lim_{k \rightarrow \infty} \frac{|x_k^* - y_k^*|}{|x_k - y_k|} = \frac{1}{\text{dist}(\mathcal{A}, 0)}.$$

By Case 1, Case 2 and Corollary 3.4, we get $\text{Lip}(\Pi) = 1/\text{dist}(\mathcal{A}, 0)$.

Assume now that $\text{Lip}(\Pi) = 1/\text{dist}(\mathcal{A}, 0)$. Choose two sequences of distinct points $\{x_k\}, \{y_k\} \subset \mathcal{A}$ such that

$$\text{Lip}(\Pi) = \lim_{k \rightarrow \infty} \frac{|x_k^* - y_k^*|}{|x_k - y_k|}.$$

By (3.2), we have

$$(3.11) \quad \lim_{k \rightarrow \infty} \frac{|x_k^* - y_k^*|}{|x_k - y_k|} \leq \limsup_{k \rightarrow \infty} \frac{2}{|x_k| + |y_k|} \leq 1/\text{dist}(\mathcal{A}, 0).$$

and hence

$$(3.12) \quad \limsup_{k \rightarrow \infty} \frac{2}{|x_k| + |y_k|} = 1/\text{dist}(\mathcal{A}, 0) = \lim_{k \rightarrow \infty} \frac{|x_k^* - y_k^*|}{|x_k - y_k|}.$$

There exists a subsequence still denoted by $\{x_k\}$ tending to a point h such that $\text{dist}(\mathcal{A}, 0) = |h|$. Similarly, $\{y_k\}$ tends to a point h' with $\text{dist}(\mathcal{A}, 0) = |h'|$. If $h = h'$, by (3.10) the sequences $\{x_k\}$ and $\{y_k\}$ satisfy the relation (3.6). Thus \mathcal{A} is admissible. \square

O. Martio and U. Srebro defined the so-called *radial stretching* which in fact is the inverse of the radial extension φ_1 in (1.6). They proved that

Lemma 3.13. [MS, Lemma 2.4] *Let \mathcal{M} be the boundary of a strictly starlike domain w.r.t. the origin and φ be the inverse map of the radial projection to the unit sphere. If φ is bi-Lipschitz, then φ_1 is bi-Lipschitz.*

Lemma 3.14. [MS, Lemma 2.7] *If the domain bounded by \mathcal{M} satisfies the β -cone condition for some $\beta \in (0, \pi/4]$, then φ_1 is bi-Lipschitz.*

By Lemma 3.13, it is clear that φ_1 is bi-Lipschitz if and only if φ is bi-Lipschitz.

Theorem 3.15. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin. Let $\varphi : S^{n-1} \rightarrow \mathcal{M}$ be a homeomorphism which sends $R \cap S^{n-1}$ to $R \cap \mathcal{M}$, where R is the ray from 0. Then*

$$(3.16) \quad \text{Lip}(\varphi^{-1}) = \frac{1}{\text{dist}(\mathcal{M}, 0)} = \limsup_{r \rightarrow 0} \sup_{|z - \zeta| < r} \frac{|z - \zeta|}{|\varphi(z) - \varphi(\zeta)|}.$$

Moreover, φ is bi-Lipschitz if and only if the domain bounded by \mathcal{M} satisfies the α -tangent condition. Then we have

$$(3.17) \quad \text{Lip}(\varphi) = \sup_{\zeta} \limsup_{z \rightarrow \zeta} \frac{|\varphi(z) - \varphi(\zeta)|}{|z - \zeta|}$$

and

$$(3.18) \quad \text{Lip}(\varphi) = \text{ess sup}_{x \in \mathcal{M}} \frac{|x|}{\sin \alpha(x)} \in \left[\frac{|x|_{\min}}{\sin \alpha}, \frac{|x|_{\max}}{\sin \alpha} \right].$$

Proof. By Example 3.7, Lemma 3.9 and (3.12), there exist two sequences of distinct points $\{x_k\}, \{y_k\} \subset \mathcal{M}$ such that $\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} y_k = h$ with $|h| = \text{dist}(\mathcal{M}, 0)$ and

$$\frac{1}{\text{dist}(\mathcal{M}, 0)} = \lim_{k \rightarrow \infty} \frac{|x_k^* - y_k^*|}{|x_k - y_k|}$$

Then

$$\text{Lip}(\varphi^{-1}) = \lim_{k \rightarrow \infty} \frac{|x_k^* - y_k^*|}{|x_k - y_k|} \leq \limsup_{r \rightarrow 0} \sup_{|z - \zeta| < r} \frac{|z - \zeta|}{|\varphi(z) - \varphi(\zeta)|} \leq \text{Lip}(\varphi^{-1}).$$

Hence (3.16) holds.

By Proposition 2.17 and Lemma 3.14, it is clear that φ is bi-Lipschitz if the domain bounded by \mathcal{M} satisfies the α -tangent condition.

On the other hand, we suppose that φ is bi-Lipschitz, then $\text{Lip}(\varphi) < \infty$. For two distinct points $x, z \in \mathcal{M}$. Let $\angle(z, 0, x) = \theta$, then by the Law of Sines,

$$\begin{aligned} \sin \angle(z, x, 0) &= \frac{|z| \sin \theta}{|z - x|} \\ &= \frac{|\varphi^{-1}(z) - \varphi^{-1}(x)|}{|z - x|} |z| \sin \left(\frac{\pi - \theta}{2} \right) \\ &\geq \frac{\text{dist}(\mathcal{M}, 0)}{\text{Lip}(\varphi)} \sin \left(\frac{\pi - \theta}{2} \right). \end{aligned}$$

Hence

$$\liminf_{z \rightarrow x} \sin \angle(z, x, 0) \geq \frac{\text{dist}(\mathcal{M}, 0)}{\text{Lip}(\varphi)}.$$

By Corollary 3.3 we have $\text{Lip}(\varphi) \geq \text{dist}(\mathcal{M}, 0)$ which implies that there exists $\alpha \in (0, \pi/2]$ such that $\alpha(x) = \liminf_{z \rightarrow x} \angle(z, x, 0) \geq \alpha$.

The equality (3.17) follows from (3.20).

The equality (3.18) follows from (3.21) and (4.1). \square

Lemma 3.19. [K2, Theorem 3.1] *Let γ be the curve which is the boundary of a domain in \mathbb{R}^2 strictly starlike w.r.t. the origin. Let $\phi(t) = r(t)e^{it} : [0, 2\pi] \rightarrow \gamma$ and $\varphi(e^{it}) = \phi(t)$. Then*

$$(3.20) \quad \sup_{t \neq s} \frac{|\phi(t) - \phi(s)|}{|e^{it} - e^{is}|} = \sup_s \limsup_{t \rightarrow s} \frac{|\phi(t) - \phi(s)|}{|e^{it} - e^{is}|}.$$

Moreover, if $\sup_s \limsup_{t \rightarrow s} \frac{|\phi(t) - \phi(s)|}{|e^{it} - e^{is}|} < \infty$, then

$$(3.21) \quad \text{Lip}(\varphi) = \text{Lip}(\phi) = \text{ess sup}_t \sqrt{r'^2(t) + r^2(t)}.$$

In [K2, Theorem 3.1] a condition similar to the one in Lemma was studied under the additional assumption that ϕ be Lipschitz. However, this assumption is redundant by Lemma 3.22.

Proof. If $\sup_s \limsup_{t \rightarrow s} \frac{|\phi(t) - \phi(s)|}{|e^{it} - e^{is}|} = \infty$, then it is trivial that (3.20) holds. We now consider that $\sup_s \limsup_{t \rightarrow s} \frac{|\phi(t) - \phi(s)|}{|e^{it} - e^{is}|} < \infty$. Then

$$\sup_s \limsup_{t \rightarrow s} \frac{|\phi(t) - \phi(s)|}{|e^{it} - e^{is}|} = \sup_s \limsup_{t \rightarrow s} \frac{|\phi(t) - \phi(s)|}{|t - s|} < \infty.$$

Let $\phi(t) = (\phi_1(t), \phi_2(t))$. By Lemma 3.22 ϕ_1, ϕ_2 are both Lipschitz continuous and hence ϕ is a Lipschitz map. Then by [K2, Theorem 3.1], the equality (3.20) holds. We also have

$$\text{Lip}(\phi) \leq \text{Lip}(\varphi) = \sup_s \limsup_{t \rightarrow s} \frac{|\varphi(e^{it}) - \varphi(e^{is})|}{|e^{it} - e^{is}|} = \sup_s \limsup_{t \rightarrow s} \frac{|\phi(t) - \phi(s)|}{|t - s|} \leq \text{Lip}(\phi).$$

This inequality together with (2.13) yields (3.21). \square

Lemma 3.22. *Assume that $g : [0, 1] \rightarrow \mathbb{R}$ is a real function such that*

$$M = \sup_t \limsup_{s \rightarrow t} \frac{|g(s) - g(t)|}{|t - s|} < \infty,$$

then g is M -Lipschitz continuous.

Proof. Suppose that E is a measurable subset of an interval, g is a function on E such that for every $x \in E$, $|D^+(g)(x)| \leq M$, where

$$|D^+(g)(x)| = \limsup_{y \rightarrow x} \frac{|g(x) - g(y)|}{|x - y|}.$$

Then

$$\text{mes}(g(E)) \leq M \text{mes}(E),$$

see e.g. the proof of Lemma 3.13 in [L]. Here mes is the (outer) Lebesgue measure. Note that [L, Lemma 3.13] assumes that g is differentiable, but the proof uses only the upper bound $|D^+(g)(x)| \leq M$ for all x . In our case, E is an interval $[s, t]$ in $[0, 1]$ and

$$M = \sup_x \limsup_{y \rightarrow x} \frac{|g(x) - g(y)|}{|x - y|}.$$

Then for every $s, t \in [0, 1]$, we have

$$|g(s) - g(t)| \leq \text{mes}(g([s, t])) \leq M|s - t|.$$

Hence g is M -Lipschitz. \square

By (3.16), we have

Corollary 3.23. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin which satisfies the α -tangent condition. Let φ be as in Theorem 3.15. Then*

$$\liminf_{r \rightarrow 0} \inf_{|z - \zeta| < r} \frac{|\varphi(z) - \varphi(\zeta)|}{|z - \zeta|} = \inf_{z, \zeta \in S^{n-1}} \frac{|\varphi(z) - \varphi(\zeta)|}{|z - \zeta|} = \text{dist}(\mathcal{M}, 0).$$

4. BI-LIPSCHITZ AND QUASICONFORMAL CONSTANTS FOR RADIAL EXTENSIONS

For the statement of our results we carry out some preliminary considerations. Let γ be the boundary of a domain in \mathbb{R}^2 strictly starlike w.r.t. the origin which satisfies the α -tangent condition. We will recall some properties of γ . Let $t \mapsto r(t)e^{it}$ be the polar parametrization of γ . If the curve γ is smooth, following the notations in [K1], the angle α_t between $\zeta = r(t)e^{it}$ and the positive oriented tangent at ζ satisfies

$$(4.1) \quad \cot \alpha_t = \frac{r'(t)}{r(t)}.$$

Observe that for the curve γ , we have

$$0 < \alpha_1 = \inf_t \alpha_t \leq \frac{\pi}{2} \leq \sup_t \alpha_t = \alpha_2 < \pi.$$

Then

$$(4.2) \quad \alpha = \min\{\alpha_1, \pi - \alpha_2\}.$$

Let $z = \rho e^{it}, \varphi(e^{it}) = r(t)e^{it}$. Let $f^g(z) = g(\rho)\varphi(e^{it})$, for some real positive smooth increasing function g , with $g(0) = 0$ and $g(1) = 1$. By direct calculation,

$$|f^g_z| = \frac{1}{2|z|} |g' \cdot r \cdot \rho - ig \cdot (r' + ir)| \quad \text{and} \quad |f^g_{\bar{z}}| = \frac{1}{2|z|} |g' \cdot r \cdot \rho + ig \cdot (r' + ir)|,$$

by (4.1), we have

$$|f^g_z| = \frac{r}{2|z|} |g' \cdot \rho + g - ig \cot \alpha_t| \quad \text{and} \quad |f^g_{\bar{z}}| = \frac{r}{2|z|} |g' \cdot \rho - g + ig \cot \alpha_t|.$$

In order to minimize the constant of quasiconformality we define the function

$$\kappa(g, z) := \mu_{f^g}^2(z).$$

Then

$$\kappa(g, z) = \frac{|g' \cdot \rho - g + ig \cot \alpha_t|^2}{|g' \cdot \rho + g - ig \cot \alpha_t|^2} = \frac{(h-1)^2 + \cot^2 \alpha_t}{(h+1)^2 + \cot^2 \alpha_t},$$

where

$$h(\rho) = \frac{g'(\rho)\rho}{g(\rho)}.$$

Since

$$\kappa(g, z) \leq \frac{(h-1)^2 + \cot^2 \alpha}{(h+1)^2 + \cot^2 \alpha},$$

we easily see that the derivative of the last expression w.r.t. h is

$$\frac{4(h - \csc \alpha)(h + \csc \alpha)}{((h+1)^2 + \cot^2 \alpha)^2},$$

which means that the minimum of the expression

$$\frac{(h-1)^2 + \cot^2 \alpha}{(h+1)^2 + \cot^2 \alpha}$$

is attained for $h = \csc \alpha$. Further the unique solution of differential equation

$$\frac{g'(\rho)\rho}{g(\rho)} = \csc \alpha$$

with $g(0) = 0$ and $g(1) = 1$ is $g(\rho) = \rho^{\csc \alpha}$. This means that the minimal constant of quasiconformality for radial stretching mappings is attained by the mapping

$$\varphi^\circ(z) = |z|^{\csc \alpha} \varphi(z/|z|).$$

Further

$$(4.3) \quad \Lambda_{\varphi^\circ}(z) = |\varphi_z^\circ| + |\varphi_{\bar{z}}^\circ| = \frac{r(t)\rho^{\csc \alpha - 1}}{2} (|\csc \alpha + 1 - i \cot \alpha_t| + |\csc \alpha - 1 + i \cot \alpha_t|)$$

and

$$(4.4) \quad \lambda_{\varphi^\circ}(z) = |\varphi_z^\circ| - |\varphi_{\bar{z}}^\circ| = \frac{r(t)\rho^{\csc \alpha - 1}}{2} (|\csc \alpha + 1 - i \cot \alpha_t| - |\csc \alpha - 1 + i \cot \alpha_t|)$$

or

$$(4.5) \quad \frac{1}{\lambda_{\varphi^\circ}(z)} = \frac{\rho^{1 - \csc \alpha}}{2r(t) \csc \alpha} (|\csc \alpha + 1 - i \cot \alpha_t| + |\csc \alpha - 1 + i \cot \alpha_t|).$$

This implies that φ° is quasiconformal with $k = \tan(\frac{\pi}{4} - \frac{\alpha}{2})$ and $K = \cot \frac{\alpha}{2}$.

In general for $a > 0$ we define $\varphi_a(z) = |z|^a \varphi(e^{it})$ and obtain

$$(4.6) \quad \Lambda_{\varphi_a}(z) = \frac{r(t)\rho^{a-1}}{2}(|a+1-i\cot\alpha_t| + |a-1+i\cot\alpha_t|)$$

and

$$(4.7) \quad \frac{1}{\lambda_{\varphi_a}(z)} = \frac{\rho^{1-a}}{2r(t)a}(|a+1-i\cot\alpha_t| + |a-1+i\cot\alpha_t|).$$

We now formulate the main result of this section.

Theorem 4.8. *Let γ be the boundary of a domain in \mathbb{R}^2 strictly starlike w.r.t. the origin, and with a polar parametrization by a homeomorphism $\varphi(e^{it}) = r(t)e^{it} : S^1 \rightarrow \gamma$. Let $a > 0$ and $\varphi_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the radial extension of φ with $\varphi_a(z) = |z|^a \varphi(z/|z|)$ and $\varphi_a(0) = 0$. Then*

- a) φ_a is bi-Lipschitz if and only if the domain bounded by γ satisfies the α -tangent condition and $a = 1$. Moreover,

$$(4.9) \quad \text{Lip}(\varphi_1) = L_1 = \frac{|r|_{\max}}{2} \left(\sqrt{\csc^2 \alpha + 3} + \sqrt{\csc^2 \alpha - 1} \right)$$

and

$$(4.10) \quad \text{Lip}(\varphi_1^{-1}) = L_2 = \frac{1}{2|r|_{\min}} \left(\sqrt{\csc^2 \alpha + 3} + \sqrt{\csc^2 \alpha - 1} \right).$$

For the bi-Lipschitz constant $L = \max\{L_1, L_2\}$ we have

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} L = \max \left\{ \text{dist}(\gamma, 0), \text{dist}(\gamma, 0)^{-1} \right\}.$$

- b) φ_a is K_a -quasiconformal if and only if the domain bounded by γ satisfies the α -tangent condition. The constant of quasiconformality is

$$(4.11) \quad K_a = \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha} + \sqrt{(1+a)^2 + \cot^2 \alpha} \right)^2.$$

The minimal constant of quasiconformality is attained by $\varphi^\circ(z) = |z|^{\csc \alpha} \varphi(z/|z|)$ with $\varphi^\circ(0) = 0$ and for this mapping we have

$$k = \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right), \quad K = \cot \frac{\alpha}{2}.$$

Proof. a). If φ_a is bi-Lipschitz then we have $a = 1$ by (2.13) and (4.6). By Lemma 3.13 and Theorem 3.15 we conclude that φ_1 is bi-Lipschitz if and only if the domain bounded by γ satisfies the α -tangent condition.

The Lipschitz constants L_1, L_2 follow from (2.8), (2.9), (4.6) (4.7), and (2.13).

To show $\lim_{\alpha \rightarrow \frac{\pi}{2}} L = \max \left\{ \text{dist}(\gamma, 0), \text{dist}(\gamma, 0)^{-1} \right\}$, we use (4.1). Then

$$\log(r(t)) - \log(r(0)) = \int_0^t \cot \alpha_s ds.$$

Without loss of generality we may assume that $r(0) = \text{dist}(\gamma, 0)$. Then

$$r(t) \leq \text{dist}(\gamma, 0) \exp(t \cot \alpha),$$

and therefore

$$\text{dist}(\gamma, 0) \leq r(t) \leq \text{dist}(\gamma, 0) \exp(2\pi \cot \alpha) \rightarrow \text{dist}(\gamma, 0), \quad \alpha \rightarrow \frac{\pi}{2}.$$

Thus by (4.9) and (4.10) we have

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} L = \max \{ \text{dist}(\gamma, 0), \text{dist}(\gamma, 0)^{-1} \}.$$

b). If the domain bounded by γ satisfies the α -tangent condition, then φ_1 is bi-Lipschitz by a). Let $R(1/r, r) = B^2(r) \setminus \overline{B^2}(1/r)$, $r > 1$. The function $g(z) = |z|^{a-1}$ is locally Lipschitz in $\overline{R(1/r, r)}$ and hence g is ACL and a.e. differentiable in $R(1/r, r)$. Therefore we have that $\varphi_a = g \cdot \varphi_1$ is ACL and a.e. differentiable in $R(1/r, r)$. Moreover, by (4.6) and (4.7), for $x \in R(1/r, r)$

$$H(\varphi'_a(x)) \leq K_a = \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha} + \sqrt{(1+a)^2 + \cot^2 \alpha} \right)^2,$$

which implies that φ_a is K_a -quasiconformal in $R(1/r, r)$. Letting $r \rightarrow \infty$, we see that φ_a is K_a -quasiconformal in $\mathbb{R}^2 \setminus \{0\}$. Since an isolated boundary point is removable singularity, we obtain that φ_a is K_a -quasiconformal in \mathbb{R}^2 .

We now prove the reverse implication. We have to show that if φ_a is K_a -quasiconformal then the domain bounded by γ satisfies the α -tangent condition. We know that if φ_a is K_a -quasiconformal, then φ_a is ACL and differentiable a.e. and hence $\phi(t) = \varphi(e^{it})$ is absolutely continuous and a.e. differentiable. By virtue of (4.6) and (4.7), the domain bounded by γ satisfies the α -tangent condition a.e., i.e. for a.e. t

$$\begin{aligned} \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha_t} + \sqrt{(1+a)^2 + \cot^2 \alpha_t} \right)^2 \\ \leq K_a = \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha} + \sqrt{(1+a)^2 + \cot^2 \alpha} \right)^2 \end{aligned}$$

implying that $\alpha_t \geq \alpha$ for a.e. t . Here α is chosen so that formula (4.11) holds. Such a value $\alpha > 0$ exists because

$$\lim_{\alpha \rightarrow 0} \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha} + \sqrt{(1+a)^2 + \cot^2 \alpha} \right)^2 = \infty$$

and

$$\lim_{\alpha \rightarrow \pi/2} \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha} + \sqrt{(1+a)^2 + \cot^2 \alpha} \right)^2 = \frac{1}{4a} (|1-a| + |1+a|)^2.$$

Without loss of generality, we may assume that $\phi(0) = 1$ is the non-smooth point and let $\beta = \liminf_{z \rightarrow 1} \alpha(z, 1)$ (recall that $\alpha(z, 1)$ is the acute angle between $z - 1$ and 1). Since ϕ is absolutely continuous, we have

$$\phi(t) = 1 + \int_0^t \psi(s) ds, \quad \psi \in L^1([0, 2\pi])$$

and

$$\phi'(t) = \psi(t) \text{ a.e.}$$

Let $\phi(s) \in \gamma$ with $s \in (0, t)$ be the smooth point and $t \in (0, \pi/2)$. Let θ be the acute angle between $[0, 1]$ and $[\phi(t), 1]$. Let $\beta(s) = \arg \psi(s)$ and α_s be the acute angle between $[0, \phi(s)]$ and tangent line of γ at $\phi(s)$. Then

$$(4.12) \quad \beta(s) = s + \alpha_s \quad \text{or} \quad \beta(s) = s + \pi - \alpha_s$$

and

$$(4.13) \quad \cos \theta = \frac{|\operatorname{Re}(\phi(t) - \phi(0))|}{|\phi(t) - \phi(0)|} = \frac{|\int_0^t \cos \beta(s) |\psi(s)| ds|}{\sqrt{\left(\int_0^t \cos \beta(s) |\psi(s)| ds\right)^2 + \left(\int_0^t \sin \beta(s) |\psi(s)| ds\right)^2}}.$$

Now we consider the quantity

$$A(t) = \frac{|\int_0^t \sin \beta(s) |\psi(s)| ds|}{|\int_0^t \cos \beta(s) |\psi(s)| ds|}.$$

By (4.12) we have $\sin \beta(s) = \sin(\alpha_s + s)$ or $\sin \beta(t) = \sin(\alpha_s - s)$ and $\cos \beta(s) = \cos(\alpha_s + s)$ or $\cos \beta(s) = -\cos(\alpha_s - s)$. Then we have

$$A(t) = \frac{|\int_0^t \sin(\alpha_s + s) |\psi(s)| ds|}{|\int_0^t \cos(\alpha_s + s) |\psi(s)| ds|}$$

or

$$A(t) = \frac{|\int_0^t \sin(\alpha_s - s) |\psi(s)| ds|}{|\int_0^t \cos(\alpha_s - s) |\psi(s)| ds|}.$$

We now use the fact that $\alpha_s \geq \alpha$. Then for small enough t (depending on α) we obtain

$$A(t) \geq B(t) = \frac{|\int_0^t \sin(\alpha \pm s) |\psi(s)| ds|}{|\int_0^t \cos(\alpha \pm s) |\psi(s)| ds|} \geq \frac{|\sin(\alpha - t)| \int_0^t |\psi(s)| ds}{|\cos(\alpha - t)| \int_0^t |\psi(s)| ds} = \tan(\alpha - t).$$

Since

$$\cos \theta = \frac{1}{\sqrt{1 + A(t)^2}},$$

we obtain that

$$\liminf_{t \rightarrow 0} \cos \theta = \liminf_{t \rightarrow 0} \frac{1}{\sqrt{1 + A(t)^2}} \leq \liminf_{t \rightarrow 0} \frac{1}{\sqrt{1 + B(t)^2}} = \cos \alpha.$$

Thus $\beta \geq \alpha$ as desired. The proof of the last statement of b) follows from the considerations carried out before the formulation of the theorem. \square

We now generalize Theorem 4.8 to the n -dimensional case.

Theorem 4.14. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n ($n \geq 3$) strictly starlike w.r.t. the origin and with a polar parametrization by a homeomorphism $\varphi(x) = r(x)x : S^{n-1} \rightarrow \mathcal{M}$. Let $\varphi_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the radial extension of φ , defined by $\varphi_a(x) = |x|^a \varphi(x/|x|)$ i.e. $\varphi_a(x) = |x|^{a-1} R(x)x$, and $\varphi_a(0) = 0$, where $R(x) = r(x/|x|)$ is a positive real function. Then*

- a) φ_a is bi-Lipschitz if and only if the domain bounded by \mathcal{M} satisfies the α -tangent condition and $a = 1$. Moreover,

$$(4.15) \quad \operatorname{Lip}(\varphi_1) = L_1 = \frac{|r|_{\max}}{2} \left(\sqrt{\csc^2 \alpha - 1} + \sqrt{\csc^2 \alpha + 3} \right)$$

and

$$(4.16) \quad \operatorname{Lip}(\varphi_1^{-1}) = L_2 = \frac{1}{|r|_{\min}} \left(\sqrt{\csc^2 \alpha - 1} + \sqrt{\csc^2 \alpha + 3} \right).$$

For the bi-Lipschitz constant

$$(4.17) \quad L = \max\{L_1, L_2\}$$

and

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} L = \max \{ \text{dist}(\mathcal{M}, 0), \text{dist}(\mathcal{M}, 0)^{-1} \}.$$

- b) φ_a is K_a -quasiconformal if and only if the domain bounded by \mathcal{M} satisfies the α -tangent condition. The constant of quasiconformality is

$$K_a = \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha} + \sqrt{(1+a)^2 + \cot^2 \alpha} \right)^2.$$

The minimal constant of quasiconformality is attained for $a = \csc \alpha$ and the mapping $\varphi^\circ(z) = |z|^{\csc \alpha} \varphi(z/|z|)$ with $\varphi^\circ(0) = 0$ is K -quasiconformal with

$$K = \cot \frac{\alpha}{2}.$$

Proof. a) For every $x, y \in \mathbb{R}^n$, the points $0, x, y, \varphi_a(x), \varphi_a(y)$ are in the same plane. Therefore by using the similar argument as in Theorem 4.8 a), we see that the α -tangent condition of the domain bounded by \mathcal{M} is equivalent to the bi-Lipschitz continuity of the radial extension φ_1 .

b) Assume now that the domain bounded by \mathcal{M} satisfies the α -tangent condition. By an argument similar to Theorem 4.8 b) we have that φ_a is quasiconformal in $R(1/r, r) = B^n(r) \setminus \overline{B}^n(1/r) \supset \mathcal{M}$. Then it is differentiable a.e. in $R(1/r, r)$. Let $x = (x_1, \dots, x_n) \in R(1/r, r) \setminus E$, where the Lebesgue measure $|E| = 0$, and let $e_k, k = 1, \dots, n$ denote the standard orthonormal basis and let us find $\lambda_{\varphi_a}(x)$ and $\Lambda_{\varphi_a}(x)$. Since

$$\Lambda_{\varphi_a}(x) = \sup_{|h|=1} |\varphi'_a(x)h|$$

and S^{n-1} is compact, there exists $h \in S^{n-1}$ such that $\Lambda_{\varphi_a}(x) = |\varphi'_a(x)h|$. Let Σ be the 2-dimensional plane passing through $0, x, h$ and let Σ' be another 2-dimensional plane passing through $0, \varphi_a(x), \varphi'_a(x)h$. Since $\varphi_a(x) = |x|^{a-1}R(x)x$, we get

$$\varphi'_a(x)h = |x|^{a-3} (\langle \text{grad} R(x) | h \rangle |x|^2 + (a-1) \langle x | h \rangle R(x)) x + |x|^{a-1} R(x)h,$$

which implies that Σ' can be chosen to be equal to Σ . Let T be an orthogonal transformation which maps the plane $\mathbb{C}' = \{(x_1, x_2, 0, \dots, 0) : x_1 + ix_2 \in \mathbb{C}\} \cong \mathbb{C}$ onto Σ such that $T(|x|e_1) = x$ and $T(\cos \theta e_1 + \sin \theta e_2) = h$. Here θ satisfies $|x| \cos \theta = \langle x | h \rangle$. The mapping T is a linear isometry of \mathbb{R}^n . Define

$$\tilde{\varphi}_a(y_1, y_2) = P(T^{-1} \varphi_a(T(y_1, y_2, 0, \dots, 0))),$$

where $P : \mathbb{C}' \rightarrow \mathbb{C}$ is the isometry $P(z) = (z_1, z_2)$.

Then $\tilde{\varphi}_a(|x|, 0) = PT^{-1}(\varphi_a(x))$ and

$$\begin{aligned} |\tilde{\varphi}'_a(|x|, 0)| &= \sup_{\beta} |\tilde{\varphi}'_a(|x|, 0)(\cos \beta, \sin \beta)| \\ &= \sup_{\beta} |(PT^{-1} \cdot \varphi'_a(x) \cdot T)(\cos \beta e_1 + \sin \beta e_2)| \leq |\varphi'_a(x)|. \end{aligned}$$

By choosing $\beta = \theta$, we see that

$$|(PT^{-1} \cdot \varphi'_a(x) \cdot T)(\cos \beta e_1 + \sin \beta e_2)| = |\varphi'_a(x)h| = |\varphi'_a(x)|,$$

which implies $|\tilde{\varphi}'_a(|x|, 0)| = |\varphi'_a(x)|$. By making use of the proof of two dimensional case (4.6), we obtain that

$$\Lambda_{\varphi_a}(x) = \Lambda_{\tilde{\varphi}_a}(|x|, 0) = \frac{r(x)|x|^{a-1}}{2} \left(\sqrt{(a-1)^2 + \cot^2 \alpha_{x,h}} + \sqrt{(1+a)^2 + \cot^2 \alpha_{x,h}} \right).$$

Here $\alpha_{x,h}$ is the acute angle between the tangent line on $\mathcal{M} \cap \Sigma$ at $\varphi(x/|x|)$ and the vector $\varphi_a(x)$.

Similar arguments and (4.7) yield that

$$\lambda_{\varphi_a}(x) = \lambda_{\tilde{\varphi}_a}(|x|, 0) = \frac{2ar(x)}{|x|^{1-a}} \left(\sqrt{(a-1)^2 + \cot^2 \alpha_{x,h'}} + \sqrt{(1+a)^2 + \cot^2 \alpha_{x,h'}} \right)^{-1},$$

where the angle $\alpha_{x,h'}$ is possibly different from $\alpha_{x,h}$. Thus we have

$$\begin{aligned} H(\varphi'_a(x)) &= \frac{\Lambda_f(x)}{\lambda_f(x)} = \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha_{x,h}} + \sqrt{(1+a)^2 + \cot^2 \alpha_{x,h}} \right) \\ &\quad \cdot \left(\sqrt{(a-1)^2 + \cot^2 \alpha_{x,h'}} + \sqrt{(1+a)^2 + \cot^2 \alpha_{x,h'}} \right). \end{aligned}$$

Hence f is K_a -quasiconformal in $R(1/r, r)$ with

$$K_a = \frac{1}{4a} \left(\sqrt{(a-1)^2 + \cot^2 \alpha} + \sqrt{(1+a)^2 + \cot^2 \alpha} \right)^2.$$

Letting $r \rightarrow \infty$, we have that φ_a is K_a -quasiconformal in $\mathbb{R}^n \setminus \{0\}$ and hence in \mathbb{R}^n .

On the other hand, if φ_a is K_a -quasiconformal in \mathbb{R}^n , then by Theorem 4.8 b) we obtain that the domain bounded by \mathcal{M} satisfies the α -tangent condition.

For $a = \csc \alpha$, applying the formula of K_a to $\varphi^\circ(z) = |z|^{\csc \alpha} \varphi(z/|z|)$, we obtain $K = \cot \alpha/2$, and this is the minimal constant of quasiconformality.

This completes the proof. \square

Example 4.18. a) Let $\mathcal{M} = \partial D$, where D is the cone $\{(x, y, z) : (z-2)^2/3 = x^2 + y^2, -1 \leq z \leq 2\}$. Then $\alpha = \pi/6$ and hence $H(\varphi^\circ) = 2 + \sqrt{3} \approx 3.73$.

b) Let $\mathcal{M} = \partial D$, where D is the cylinder $\{(x, y, z) : x^2 + y^2 = 1, -1 \leq z \leq 1\}$. Then $\alpha = \pi/4$ and hence $H(\varphi^\circ) = \sqrt{2} + 1 \approx 2.41$.

c) Let $\mathcal{M} = \partial D$, where $D = [-1, 1]^3$ is the cube. Then $\sin \alpha = 1/\sqrt{3}$ and hence $H(\varphi^\circ) = \sqrt{3} + \sqrt{2} \approx 3.15$.

d) Let $\mathcal{M} = \partial D$, where D is the ellipsoid $\{(x_1, \dots, x_n) : (x_1/a_1)^2 + \dots + (x_n/a_n)^2 \leq 1, 0 < a_1 \leq \dots \leq a_n\}$. We first consider the case of the ellipse $\{(x, y) : (x/a)^2 + (y/b)^2 \leq 1, 0 < a < b\}$ whose polar parametrization is

$$r(t) = \frac{ab}{\sqrt{b^2 \cos^2 t + a^2 \sin^2 t}}, \quad t \in [0, 2\pi].$$

By using the argument of symmetry it suffices to consider $t \in [0, \pi/2]$. By (4.1), we have

$$0 \leq \cot \alpha_t = \frac{r'(t)}{r(t)} = \frac{b^2 - a^2}{b^2 \frac{\cos t}{\sin t} + a^2 \frac{\sin t}{\cos t}} \leq \frac{b^2 - a^2}{2ab}.$$

The equality holds if and only if $\tan t = b/a > 1$. Then

$$\cot \frac{\alpha_t}{2} = \sqrt{1 + \cot^2 \alpha_t} + \cot \alpha_t \leq \frac{b}{a}.$$

Therefore for the ellipsoid the angle α is minimized in the ellipse $(x/a_1)^2 + (y/a_n)^2 \leq 1$ and its value is $\alpha = 2 \arctan(a_1/a_n)$ and $H(\varphi^\circ) = a_n/a_1$. Since the linear dilatation of linear mapping $L(x_1, \dots, x_n) = (a_1 x_1, \dots, a_n x_n)$ is as well equal to a_n/a_1 , one may expect that this is the best possible constant of quasiconformality for the ellipsoid ($n \geq 3$). But this is not the case. Concerning this problem we refer to the paper of Anderson [An].

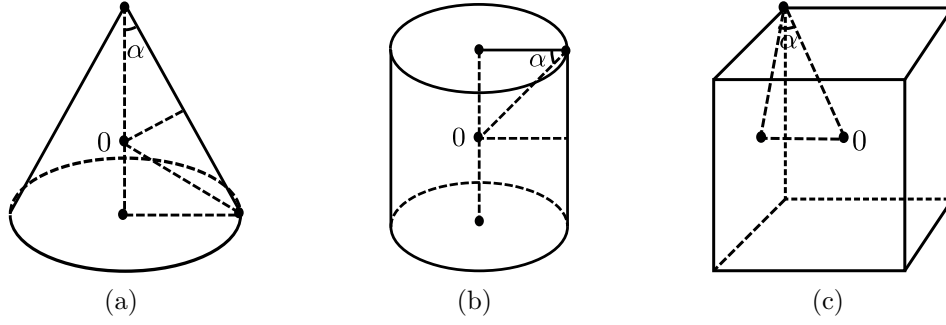


FIGURE 3. Example 4.18: (a) cone (b) cylinder (c) cube

Remark 4.19. In [MS, Lemma 2.7 and Corollary 2.8] the authors prove that the radial stretching mapping is bi-Lipschitz and quasiconformal provided that the domain bounded by \mathcal{M} satisfies the β -cone condition. Our theorem is somehow optimal, since we have concrete and approximately sharp bi-Lipschitz and quasiconformal constants. Further in [GV, Theorem 5.1] F.W. Gehring and J. Väisälä obtained some explicit estimates of K_I and K_O for a domain D satisfying the α -tangent condition in the three dimensional case. Here $K_I = \inf K_I(f)$ and $K_O = \inf K_O(f)$, where f runs through q.c. mappings of the domain D onto the unit ball $\mathbb{B}^3 \subset \mathbb{R}^3$. Indeed they proved that

$$(4.20) \quad K_I \leq 2^{-1/2} \cot \frac{\alpha}{2} \csc \frac{\alpha}{2},$$

$$(4.21) \quad K_O \leq 2^{1/2} \cot \frac{\alpha}{2} \cos \frac{\alpha}{2}.$$

Furthermore (4.20) and (4.21) are obtained by making use of a mapping which is in fact the inverse of our mapping φ° in Theorem 4.14. This implies that a part of the statement of Theorem 4.14 is not new, at least for three dimensional case. Let $H(f) = \text{ess sup}_x H(f'(x))$ and $H = \inf H(f)$, where f runs through q.c. mappings of the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ onto the domain D , then we have that

$$(4.22) \quad H \leq H(\varphi^\circ) = \cot \frac{\alpha}{2}.$$

By (4.22), we can also easily conclude that

Corollary 4.23. *Let $D \subset \mathbb{R}^n$ be the convex domain with $B^n(0, a) \subset D \subset B^n(0, b)$, $0 < a < b$. Then*

$$H \leq H(\varphi^\circ) = \frac{b + \sqrt{b^2 - a^2}}{a} < 2 \cdot \frac{b}{a}.$$

5. BI-LIPSCHITZ AND QUASICONFORMAL CONSTANTS FOR QUASI-INVERSIONS

In this section, we obtain the bi-Lipschitz constants of the quasi-inversion mappings w.r.t. the chordal metric, the Möbius metric and Ferrand's metric by using the bi-Lipschitz constants of the radial extension maps. We also obtain asymptotically sharp constants of quasiconformality of quasi-inversions. In order to explain sharpness we make the following definition.

Definition 5.1. For $t \in [0, 1]$, let \mathcal{M}_t be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin which satisfies the $\alpha_{\mathcal{M}_t}$ -tangent condition. Let $\varphi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the radial extension which sends the unit sphere to \mathcal{M}_t . We say that \mathcal{M}_t smoothly converges to the

sphere $S^{n-1}(r)$, when t goes to 0, if $\lim_{t \rightarrow 0} \operatorname{ess\,sup}_{|x|=1} |\varphi'_t(x) - r\mathbf{I}| = 0$. This in particular means that $\lim_{t \rightarrow 0} \alpha_{\mathcal{M}_t} = \pi/2$. Here \mathbf{I} is the identity matrix.

Lemma 5.2. *Let $a > 0$ and \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin and $\varphi : S^{n-1} \rightarrow \mathcal{M}$ be the homeomorphism which sends $R \cap S^{n-1}$ to $R \cap \mathcal{M}$, R is the ray from 0. Let φ_a be the radial extension of φ and $f_{\mathcal{M}}$ be the quasi-inversion in \mathcal{M} . Then $f_{\mathcal{M}} = \varphi_a \circ f_{S^{n-1}} \circ \varphi_a^{-1}$.*

Proof. It suffices to show that $f_{\mathcal{M}} \circ \varphi_a = \varphi_a \circ f_{S^{n-1}}$. For $z \neq 0$ and $z \neq \infty$ we have

$$f_{\mathcal{M}}(\varphi_a(z)) = |\varphi(z/|z|)|^2 \frac{\varphi_a(z)}{|\varphi_a(z)|^2} = \frac{\varphi(z/|z|)}{|z|^a} = \varphi_a(f_{S^{n-1}}(z)).$$

If $z = 0$ or $z = \infty$, by the definition of φ_a and $f_{\mathcal{M}}$ we still have $f_{\mathcal{M}}(\varphi_a(z)) = \varphi_a(f_{S^{n-1}}(z))$. This completes the proof. \square

By Theorem 4.14, Lemma 5.2 and (1.3), we immediately obtain the following result:

Theorem 5.3. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin which satisfies the α -tangent condition. Let $f_{\mathcal{M}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the quasi-inversion in \mathcal{M} . Then for all $x, y \in \mathbb{R}^n \setminus \{0\}$*

$$(5.4) \quad \frac{1}{L^4} \frac{|x - y|}{|x||y|} \leq |f_{\mathcal{M}}(x) - f_{\mathcal{M}}(y)| \leq L^4 \frac{|x - y|}{|x||y|},$$

where L is of the form as in (4.17) of Theorem 4.14. In particular, if $\mathcal{M} = S^{n-1}$, then (5.4) reduces to the equality

$$(5.5) \quad |f_{S^{n-1}}(x) - f_{S^{n-1}}(y)| = \frac{|x - y|}{|x||y|},$$

which is the same as (1.3) by taking $a = 0$ and $r = 1$.

Theorem 5.6. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin. Let $f_{\mathcal{M}}(z) = \frac{r_z^2 z}{|z|^2}$, $z \in \mathbb{R}^n \setminus \{0\}$, be the quasi-inversion in \mathcal{M} and let $x, y \in \mathbb{R}^n \setminus \{0\}$ with $|x| \leq |y|$. Then with $\lambda = \frac{|f_{\mathcal{M}}(y)| + |f_{\mathcal{M}}(x) - f_{\mathcal{M}}(y)|}{|f_{\mathcal{M}}(y)|}$ and $z = \lambda x$ we have*

$$(5.7) \quad |f_{\mathcal{M}}(x) - f_{\mathcal{M}}(z)| \leq |f_{\mathcal{M}}(x) - f_{\mathcal{M}}(y)| \leq (2r_y^2/r_x^2 + 1) |f_{\mathcal{M}}(x) - f_{\mathcal{M}}(z)|.$$

In particular, if $\mathcal{M} = S^{n-1}(r)$, then $\lambda = \frac{|x| + |x - y|}{|x|}$ and (5.7) reduces to

$$(5.8) \quad |f_{S^{n-1}(r)}(x) - f_{S^{n-1}(r)}(z)| \leq |f_{S^{n-1}(r)}(x) - f_{S^{n-1}(r)}(y)| \leq 3|f_{S^{n-1}(r)}(x) - f_{S^{n-1}(r)}(z)|$$

which is the same as in [BBKV, Lemma 4.5].

Proof. By calculation, we have

$$\frac{|f_{\mathcal{M}}(x) - f_{\mathcal{M}}(y)|}{|f_{\mathcal{M}}(x) - f_{\mathcal{M}}(z)|} = \frac{\lambda}{\lambda - 1} \frac{|f_{\mathcal{M}}(x) - f_{\mathcal{M}}(y)|}{|f_{\mathcal{M}}(x)|} = \frac{|f_{\mathcal{M}}(y)| + |f_{\mathcal{M}}(x) - f_{\mathcal{M}}(y)|}{|f_{\mathcal{M}}(x)|}$$

and

$$1 \leq \frac{|f_{\mathcal{M}}(y)| + |f_{\mathcal{M}}(x) - f_{\mathcal{M}}(y)|}{|f_{\mathcal{M}}(x)|} \leq 2r_y^2/r_x^2 + 1.$$

Therefore (5.7) follows.

If $\mathcal{M} = S^{n-1}(r)$, then by (5.5) we have

$$\lambda = 1 + \frac{|f_{S^{n-1}(r)}(x) - f_{S^{n-1}(r)}(y)|}{|f_{S^{n-1}(r)}(y)|} = 1 + \frac{|x - y|}{|x|} = \frac{|x| + |x - y|}{|x|}.$$

The inequality (5.8) is clear. \square

By taking $a = 1$ in Theorem 4.14, Lemma 5.2, Proposition 2.23 and Lemma 2.24–2.26, we have the following results.

Theorem 5.9. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin which satisfies the α -tangent condition. Let $f_{\mathcal{M}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the quasi-inversion in \mathcal{M} . Then $f_{\mathcal{M}}$ is an L^6 -bi-Lipschitz map w.r.t. the chordal metric, where L is of the form as in (4.17) of Theorem 4.14. Moreover, if \mathcal{M}_t , $t \in [0, 1]$, is a family of surfaces smoothly converging to the unit sphere S^{n-1} , then the bi-Lipschitz constants L_t of the quasi-inversions $f_{\mathcal{M}_t}$ tend to 1.*

Theorem 5.10. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin which satisfies the α -tangent condition. Let $G \subsetneq \mathbb{R}^n$ be a domain and $f_{\mathcal{M}} : G \rightarrow f_{\mathcal{M}}G$ be the quasi-inversion in \mathcal{M} . Then $f_{\mathcal{M}}$ is an L^8 -bi-Lipschitz map w.r.t. both the Möbius metric and Ferrand's metric, where L is of the form as in (4.17) of Theorem 4.14. Moreover, if \mathcal{M}_t , $t \in [0, 1]$, is a family of surfaces smoothly converging to the unit sphere S^{n-1} , then the bi-Lipschitz constants L_t of the quasi-inversions $f_{\mathcal{M}_t}$ tend to 1.*

By taking $a = \csc \alpha$ in Theorem 4.14, Lemma 5.2 and the fact that the inversion w.r.t. the unit sphere is a conformal mapping we obtain

Theorem 5.11. *Let \mathcal{M} be the boundary of a domain in \mathbb{R}^n , strictly starlike w.r.t. the origin which satisfies the α -tangent condition. Let $f_{\mathcal{M}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the quasi-inversion in \mathcal{M} . Then $f_{\mathcal{M}}$ is a quasiconformal mapping with*

$$K = \cot^2 \frac{\alpha}{2}.$$

If $n = 2$, then $k = (1 - \sin \alpha)/(1 + \sin \alpha)$. Furthermore, if \mathcal{M}_t , $t \in [0, 1]$, is a family of surfaces smoothly converging to the sphere $S^{n-1}(r)$, $r > 0$, then the quasiconformality constants K_t of the quasi-inversions $f_{\mathcal{M}_t}$ tend to 1.

Remark 5.12. By using a similar approach as in the proof of Theorem 4.8, to the mapping $z \mapsto r(t)^2 e^{it}/\rho$, it can be shown that the constant $K = \cot^2 \frac{\alpha}{2}$ is sharp, and the α -tangent condition is necessary as well for K -quasiconformal behavior of quasi-inversions.

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